INDIAN MARITIME UNIVERSITY

(A Central University, Government of India)

END SEMESTER EXAMINATION-DECEMBER 2019

B.Sc(Nautical Science)

Semester – I

Nautical Mathematics

(UG21T4102)

Date: 12.12.2019	Max Marks: 70
Time: 3 Hrs	Pass Marks : 35

Note: Part A is compulsory.

Answer any 6 from remaining 8 questions of Part B.

PART A

(5 x 2 = 10 marks)

1. a. Find the n^{th} derivative of $x^2 \log 3x$.

b. Evaluate $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\, dx}{1+x^2+y^2}$

c. Separate into real and imaginary part of $\exp\left(5 + \frac{i\pi}{2}\right)$.

d. Express $\int_0^{\pi/2} \sqrt{\cot\theta}\ d\theta$ in terms of Gamma function.

e. Verify Rollis theorem for $f(x) = (x+2)^3(x-3)^4$ in (-2, 3)

PART B

2. a. Change the order of integration and evaluate

$$\int_0^1 \int_x^{\sqrt{x}} xy\, dy\, dx$$

b. Find the area lying between the parabola $y = x^2$ and the line $x + y = 0$ by double integration.

(5+5 marks)

3. a. Evaluate

$$\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z}\, dz\, dy\, dx$$

b. Evaluate $\int_0^\infty e^{-ax} x^{m-1} \sin bx\, dx$ in terms of Gamma functions.

(5+5 marks)

4. a. In spherical triangle PQR angle $P = 53°5'$ sides $PQ = 70°20'$ and $PR = 110°\,14'$. Calculate angle Q.

b. In spherical triangle RST side $t = 80°\,32'$, side $r = 60°\,40'$ and angle $T = 90°$. Calculate side s.

(5+5 marks)

5. a. In spherical triangle LMN angles $N = 81° \, 50'$and $L = 119° \, 7'$. Side $m = 90°$. Calculate angle M.

b. In spherical triangle PQR angles $Q = 74° \, 52'18''$and $R = 71° \, 20'$. Side$p = 49° \, 8'$. Calculate side r.

(5+5 marks)

6. a. If $y = a\cos(\log x) + b\sin(\log x)$ show that $x^2 y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$

b. If $u = \log(x^3 + y^3 + z^3 - 3xyz)$ show that $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$.

(5+5 marks)

7. a. Find the minimum value of $x^2 + y^2 + z^2$ given $ax + by + cz = p$.

b. If $Z = f(x, y)$ and $x = e^u + e^{-v}$ and $y = e^{-u} - e^v$ prove that $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$.

(5+5 marks)

8. a. If $p = CiS\,\theta$ and $q = CiS\,\emptyset$ show that $\frac{p-q}{p+q} = i\tan\left(\frac{\theta - \emptyset}{2}\right)$.

b. Find all the values of $\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{3/4}$. Also show that the continued product of these valuesare 1.

(5+5 marks)

9. a. Expand $\cos^8\theta$ in a series of cosine multiples ofθ .

b. Show that $\sin h^{-1}(\tan\theta) = \log\left(\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)\right)$

(5+5 marks)

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Answerkey

Nautical Mathematics – UG21T4102.

Semester – I : End Semester Exam – Dec. 2019.

Q.1 a) Find the nth derivative of $e^{ax}\cos^2 x \cdot \sin x$.

→ $y = e^{ax}\cos^2 x \cdot \sin x$

$y = e^{ax}\left[(1-\sin^2 x)\cdot \sin x\right]$

$y = \left[e^{ax}(\sin x - \sin^3 x)\right]$ [½ Mark]

$y = e^{ax}\cdot\sin x - e^{ax}\cdot\sin^3 x$

$y = e^{ax}\sin x - e^{ax}\left[\frac{1}{4}(3\sin x - \sin 3x)\right]$ [½ Mark]

$y = e^{ax}\sin x - \frac{3}{4}e^{ax}\sin x + \frac{e^{ax}}{4}\sin 3x$

$y = \frac{1}{4}e^{ax}\sin x + \frac{1}{4}e^{ax}\sin 3x.$

taking nth derivative of both sides

$y_n = \frac{1}{4}\left[D^n(e^{ax}\sin x) + D^n(e^{ax}\sin 3x)\right]$

$y_n = \frac{1}{4}\left[(a^2+1)^{n/2}\cdot e^{ax}\cdot\sin\left[x + n\tan^{-1}\frac{1}{a}\right]\right] + \frac{1}{4}\left[(a^2+9)^{n/2}\cdot e^{ax}\cdot\sin\left(3x + n\tan^{-1}\frac{3}{a}\right)\right]$

[1 mark]

b) Find the value of $\tanh(\log x)$ if $x = \sqrt{3}$.

→ by defn $\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ [½ Mark]

$\therefore \tanh(\log x) = \dfrac{e^{\log x} - e^{-\log x}}{e^{\log x} + e^{-\log x}}$ [½ Mark]

$= \dfrac{x - x^{-1}}{x + x^{-1}} = \dfrac{x^2-1}{x^2+1}$ put $x = \sqrt{3}$

$\therefore \tanh(\log x) = \frac{3-1}{3+1} = \frac{2}{4} = \underline{\frac{1}{2}}$ when $x = \sqrt{3}$ [1 Mark]

c) Evaluate $\int_0^\infty x^4 e^{-x^6} dx$

Let $I = \int_0^\infty e^{-x^6} \cdot x^4 \, dx$

put $x^6 = t \Rightarrow 6x^5 dx = dt \Rightarrow x^4 dx = \frac{dt}{6 \cdot t^{1/6}}$ [1 Mark]

$I = \int_0^\infty e^{-t} \cdot \frac{dt}{6t^{1/6}}$

$I = \frac{1}{6} \int_0^\infty e^{-t} \cdot t^{-1/6} dt$

$I = \frac{1}{6} \Gamma \frac{5}{6}$ . . . . $\int_0^\infty e^{-x} x^{n-1} dx = \Gamma n$. [1 Mark]

d). Evaluate $\int_0^2 x^2 (2-x)^3 dx$

put $x = 2t$, $dx = 2dt$, $\begin{array}{c|cc} x & 0 & 2 \\ \hline t & 0 & 1 \end{array}$ (1M)

$I = \int_0^1 4t^2 \left[2 - 2t\right]^3 2dt$

$= \int_0^1 4t^2 \, 2^3 \cdot (1-t)^3 \, 2 \cdot dt$

$= 64 \int_0^1 t^2 \cdot (1-t)^3 dt$

$= 64 \, \beta(3,4)$ - - - - $\int_0^1 x^{m-1} (1-x)^{n-1} dx = \beta(m,n)$.

(1M)

e) In a spherical triangle PQR, $p = 67°$, $q = 54°$, $P = 39°$. Find angle Q, using sine formula.

→ Using sine formula, (1M)

$\frac{\sin P}{\sin p} = \frac{\sin Q}{\sin q}$

$\sin Q = 0.553099627$

$\boxed{Q = 33° 34' 47.72''}$ (1M).

Q.2 a) Change the order of integration and evaluate

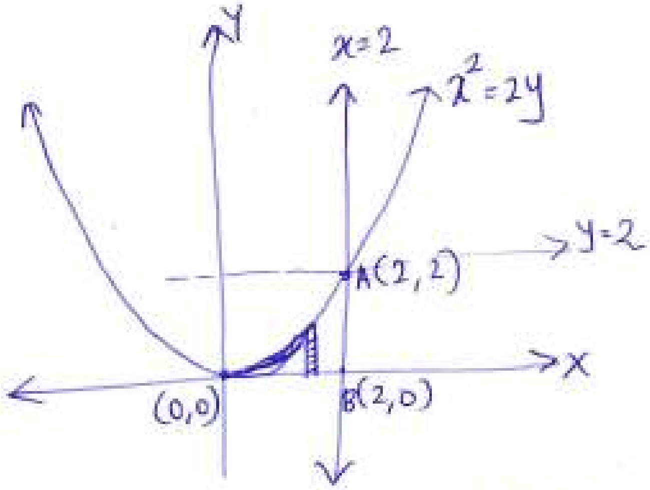
$$\int_0^2 \int_{\sqrt{2y}}^2 \frac{x^2}{\sqrt{x^4-4y^2}}\,dx\,dy$$

→ Given order is first w.r.t. x & then w.r.t. y.

a strip ||al to x-axis.

Limits are $x=\sqrt{2y}$ to $x=2$

& $y=0$ to $y=2$. (1M)



Region of integration:-

$x=\sqrt{2y}$ ie. $x^2=2y$ is a parabola symmetrical about Y-axis with vertex at the origin & opening upwards.

$x=2$ is line parallel to Y-axis.

$y=0$ is X-axis,

$y=2$ is line parallel to X-axis.

Points of intersection are A(2,2) & B(2,0) The region of integration is OAB. (1M)

Change of order of integration :-

To change order of integration consider a strip parallel to Y-axis, in the ~~reg~~ region of integration.

On this strip - y varies from 0 to $x^2/2$ & then the strip moves from $x=0$ to $x=2$.

$$\therefore I = \int_{x=0}^{x=2}\left[\int_{y=0}^{y=\frac{x^2}{2}} \frac{x^2}{\sqrt{x^4-4y^2}}\,dy\right]dx$$ (1M)

$$\Rightarrow \quad I = \int_0^2 \left[x^2 \int_0^{x^2/2} \frac{1}{\sqrt{4\left(\frac{x^4}{4} - y^2\right)}} \, dy \right] dx$$

$$I = \int_0^2 \frac{x^2}{2} \left[\int_0^{x^2/2} \frac{1}{\sqrt{\left(\frac{x^2}{2}\right)^2 - y^2}} \, dy \right] dx$$

(1M)

$$I = \int_0^2 \frac{x^2}{2} \left[\sin^{-1}\left(\frac{y}{x^2/2}\right) \right]_0^{x^2/2} dx$$

$$I = \int_0^2 \frac{x^2}{2} \left[\sin^{-1} 1 - \sin^{-1} 0 \right] dx$$

$$I = \int_0^2 \frac{x^2}{2} \left(\frac{\pi}{2} - 0\right) dx.$$

$$I = \frac{\pi}{4} \left[\frac{x^3}{3} \right]_0^2 \Rightarrow \boxed{I = \frac{2\pi}{3}}$$

(1M)

Q.2 b) Change to polar coordinates and evaluate;-

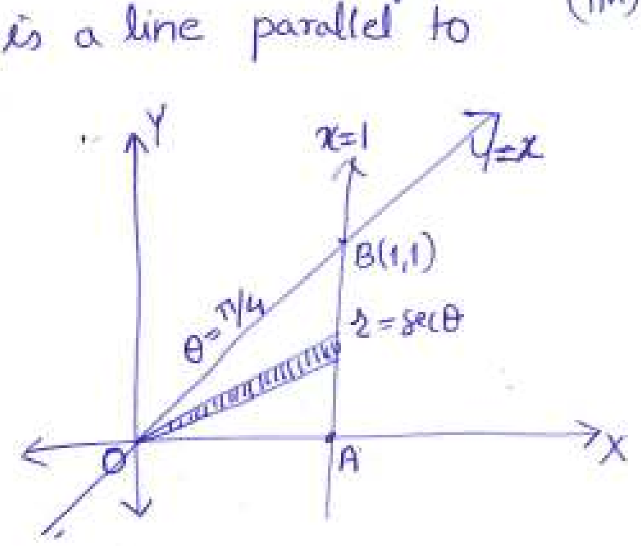
$$\int_0^1 \int_0^x (x+y) \, dy \, dx.$$

→ Region of Integration:-

$y=0$ is the X-axis; $y=x$ is a line thr origin. (1M)

$x=0$ is the Y-axis & $x=1$ is a line parallel to the y-axis.

Thus region of integration is the triangle OAB.



— Putting $x = r\cos\theta$ & $y = r\sin\theta$. the line $y = x$ becomes

$$r\sin\theta = r\cos\theta$$

ie. $\tan\theta = 1$ (1M)

$$\Rightarrow \boxed{\theta = \frac{\pi}{4}}$$

The x-axis is given by $\boxed{\theta = 0}$

The y-axis is given by $\theta = \frac{\pi}{2}$ & the line $x=1$ is given by $r\cos\theta = 1$ ie. $\boxed{r = \sec\theta}$

Considering a radial strip in the region of integration OAB, r varies from $r=0$ to $r=\sec\theta$ (1M)

& θ varies from $\theta = 0$ to $\theta = \pi/4$.

<u>Integrand</u>:— Putting $x = r\cos\theta$, $y = r\sin\theta$ in $(x+y)$ we get

$x+y = r\cos\theta + r\sin\theta = r(\cos\theta + \sin\theta)$ & $dxdy = r\,dr\,d\theta$.

$$\therefore I = \int_{\theta=0}^{\theta=\pi/4} \int_{r=0}^{r=\sec\theta} r(\cos\theta + \sin\theta) \cdot r\,dr\,d\theta.$$

$$I = \int_{\theta=0}^{\pi/4} (\cos\theta + \sin\theta)\left[\frac{r^3}{3}\right]_0^{\sec\theta} \cdot d\theta$$ (1M)

$$I = \int_0^{\pi/4} (\cos\theta + \sin\theta)\left(\frac{\sec^3\theta}{3}\right) d\theta$$

$$I = \frac{1}{3}\left[\int_0^{\pi/4} \sec^2\theta\,d\theta + \int_0^{\pi/4} \frac{\sin\theta}{\cos^3\theta}\,d\theta\right]$$

$$I = \frac{1}{3}\left\{ \left\{\tan\theta\right\}_0^{\pi/4} + \left\{\frac{1}{2\cos^2\theta}\right\}_0^{\pi/4} \right\}$$ .... (put $\cos\theta = t$)

$$I = \frac{1}{3}\left\{1 + 1 - \frac{1}{2}\right\} = \underline{\underline{\frac{1}{2}}}$$ (1M)

Q. 3a) Prove that $\beta(m,n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx$.

hence evaluate $\int_0^{\infty} \frac{x^{10}-x^{8}}{(1+x)^{30}} dx$.

$\rightarrow$ by definition,

$$\beta(m,n) = \int_0^1 x^{m-1} \cdot (1-x)^{n-1} dx \text{ ——— } ①$$

put $x = \frac{t}{1+t} \Rightarrow dx = \frac{1}{(1+t^2)} dt$

x	0	1
t	0	∞

(1M)

$\therefore$ ① becomes,

$$\beta(m,n) = \int_0^{\infty} \left(\frac{t}{1+t}\right)^{m-1} \left[1 - \frac{t}{1+t}\right]^{n-1} \cdot \frac{dt}{(1+t)^2}.$$ (1M)

$$\beta(m,n) = \int_0^{\infty} \frac{t^{m-1}}{(1+t)^{m+n}} dt$$

$$\therefore \boxed{\beta(m,n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx}$$

$\int_a^b f(x)dx = \int_a^b f(t)dt$.

(1M)

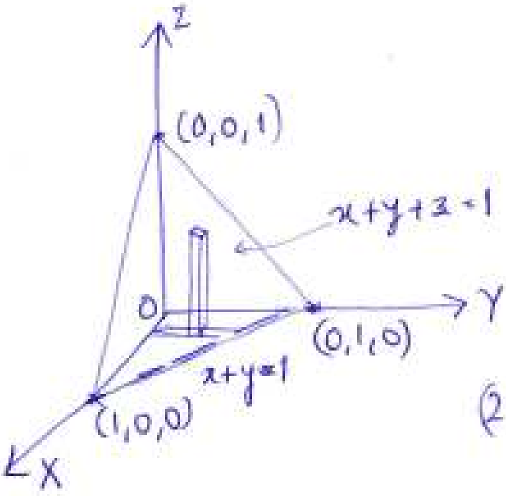
Let $I = \int_0^{\infty} \frac{x^{10}-x^{18}}{(1+x)^{30}} dx$ (1M)

$$I = \int_0^{\infty} \frac{x^{11-1}}{(1+x)^{11+19}} dx - \int_0^{\infty} \frac{x^{19-1}}{(1+x)^{19+11}} dx$$ (1M)

$$I = \beta(11, 19) - \beta(9, 11) = 0$$

Q. 3 b) Calculate the volume of the solid bounded by the planes $x=0$, $y=0$, $x+y+z=1$ & $z=0$.

→ Considering the cuboid as shown in the fig. y varies $y=0$ to $y=1-x$, x varies from $x=0$ to $x=1$ & z varies from $z=0$ to $z=1-x-y$.



(2M)

$$\therefore \text{Volume} = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz \, dy \, dx$$

$$= \int_0^1 \int_0^{1-x} \left[z \right]_0^{1-x-y} dy \, dx \qquad (1M)$$

$$= \int_0^1 \int_0^{1-x} (1-x-y) \, dy \, dx$$

$$= \int_0^1 \left[y - xy - \frac{y^2}{2} \right]_0^{1-x} . dx$$

$$= \int_0^1 \left[(1-x) - x(1-x) - \frac{(1-x)^2}{2} \right] dx \qquad (1M)$$

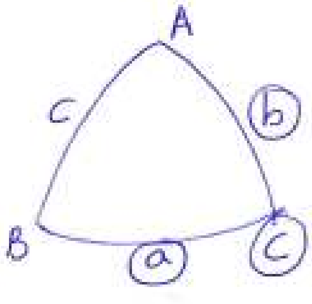
$$= \int_0^1 \left(1 - x - x + x^2 - \frac{1}{2} + \frac{2x}{2} - \frac{x^2}{2} \right) dx$$

$$= \int_0^1 \left(\frac{1}{2} - x + \frac{x^2}{2} \right) dx$$

$$= \left(\frac{1}{2}x - \frac{x^2}{2} + \frac{x^3}{6} \right)_0^1 = \frac{1}{2} - \frac{1}{2} + \frac{1}{6} = \underline{\underline{\frac{1}{6}}} \qquad (1M)$$

Q.4 a) In a spherical Δ^{le} ABC, side $a = 49°8'$, angle $C = 71°20'$, side $b = 58°23'$, Find angle A and angle B using four part formula.

⇒



To find angle A, using four part formula,

we have,

$$\cot a \cdot \sin b = \cot A \cdot \sin C + \cos C \cdot \cos b. \qquad (1M)$$

$$\Rightarrow \cot A \cdot \sin C = \cot a * \sin b - \cos C * \cos b.$$

$$\Rightarrow \cot A = \frac{\cot a \cdot \sin b - \cos C \cdot \cos b}{\sin C}$$

$$\Rightarrow \cot A = \frac{(0.736790219) - (0.1677872)}{0.947396642} \qquad (1M)$$

$$\Rightarrow \cot A = 0.600596406$$

$$\Rightarrow \boxed{A = 59° 0' 40.05''} \qquad (1/2 M)$$

To find B, using four part formula,

$$\cot b \cdot \sin a = \cot B \cdot \sin C + \cos a \cdot \cos C \qquad (1M)$$

$$\Rightarrow \cot B \cdot \sin C = \cot b \cdot \sin a - \cos a \cdot \cos C.$$

$$\Rightarrow \cot B = \frac{\cot b \cdot \sin a - \cos a \cdot \cos C}{\sin C.} \qquad (1M)$$

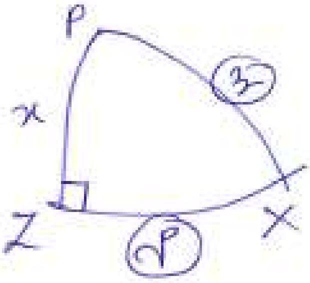
$$\Rightarrow \cot B = \frac{0.465541708 - 0.209416791}{0.947396642}$$

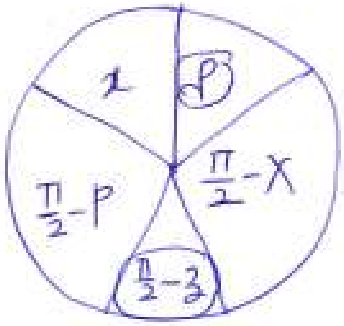
$$\Rightarrow \cot B = 0.270346025$$

$$\Rightarrow \boxed{B = 74° 52' 19.01'} \qquad (1/2 M)$$

Q. 4 b) In a spherical Δ^{le} PZX, right angled at Z, side $p = 110°20'$, side $z = 84°12'$. Find angle P, angle X and side x.

→





(2M)

Using the Napier's rules :-

① To find angle P,

$$\sin p = \cos\left(\frac{\pi}{2}-P\right)\cdot\cos\left(\frac{\pi}{2}-z\right)$$

$$\sin p = \sin P \cdot \sin z.$$

$$\sin P = \frac{\sin p}{\sin z}.$$

$$\sin P = 0.942511929$$

(1M)

$$\Rightarrow \boxed{P = 109°31'20''}$$

② To find angle X,

$$\sin\left(\frac{\pi}{2}-X\right) = \tan p \cdot \tan\left(\frac{\pi}{2}-z\right)$$

$$\cos X = \tan p \cdot \cot z$$

$$\Rightarrow \cos X = \frac{\tan p}{\tan z} = -0.274106214$$

(1M)

$$\Rightarrow \boxed{X = 105°54'31.5''}$$

③ To find side x,

$$\sin\left(\frac{\pi}{2}-z\right) = \cos x \cdot \cos p$$

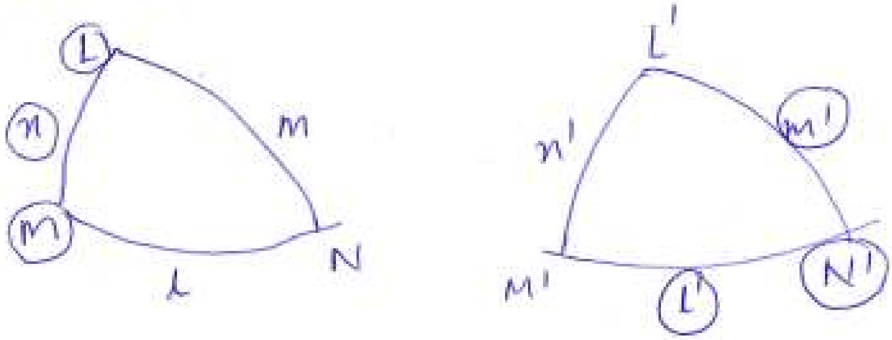
$$\cos z = \cos x \cdot \cos p$$

$$\Rightarrow \cos x = \frac{\cos z}{\cos p} = -0.290825193$$

(1M)

$$\Rightarrow \boxed{x = 106°54'26.5''}$$

Q. 5 a). In a spherical Δ^{le} LMN, $L = 88^\circ 24.5'$, $n = 100^\circ 9'$, & $M = 97^\circ 46'$. Calculate N.

→ 

(1m)

Let Δ^{le} L'M'N' be the polar triangle of Δ^{le} LMN.

∴ by supplementary theorem,

$$L + l' = 180^\circ \Rightarrow l' = 91^\circ 35' 30''$$

$$n + N' = 180^\circ \Rightarrow N' = 79^\circ 51'$$

$$M + m' = 180^\circ \Rightarrow m' = 82^\circ 14'.$$ (2m)

In Δ^{le} L'M'N', by applying cosine formula,

$$\cos n' = \cos l' \cdot \cos m' + \sin l' \cdot \sin m' \cdot \cos N'$$

$$\cos n' = 0.170788186$$ (1M)

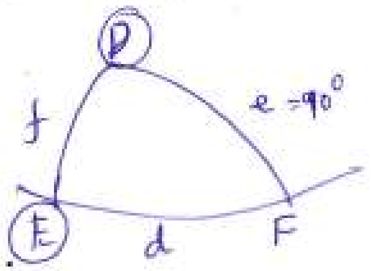
$$\Rightarrow \boxed{n' = 80^\circ 9' 58.86''}$$

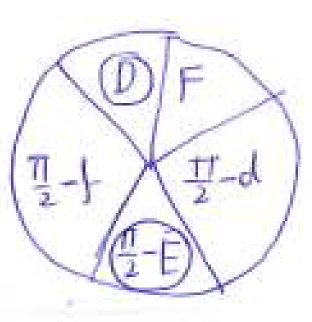
Again by supplementary thm we have,

$$n' + N = 180^\circ$$

$$\Rightarrow \boxed{N = 99^\circ 50' 1.14''}$$ (1M)

Q.5 b) In spherical Δ^{le} DEF, $D = 64^\circ 36'$, side $e = 90^\circ$ & $\angle E = 76^\circ 47'$. Calculate side d, side f, angle F.

→ 



(2m)

Using Napier's rules :-

① To find F, $\sin(\frac{\pi}{2}-E) = -\cos D \cdot \cos F$

$\cos E = -\cos D \cdot \cos F$

$\Rightarrow \cos F = -\frac{\cos E}{\cos D}$ $\Rightarrow \cos F = -0.533027129$

$\Rightarrow$ $\boxed{F = 122^\circ 12' 36.7''}$ (1M)

② To find d, $\sin D = \cos(\frac{\pi}{2}-d) \cdot \cos(\frac{\pi}{2}-E)$

$\sin D = \sin d \cdot \sin E$

$\sin d = \frac{\sin D}{\sin E} \Rightarrow \sin d = 0.927913459$

$\boxed{d = 68^\circ 6' 42.71''}$ (1M)

③ To find f, $\sin(\frac{\pi}{2}-f) = \tan D \cdot \tan(\frac{\pi}{2}-E)$

$\cos f = \tan D \cdot \cot E$

$\cos f = 0.494603047$

$\boxed{f = 119^\circ 38' 36.8''}$ (1M)

Q. 6 a) Expand $\sin^7\theta$ in a series of sines of multiples of θ.

→ let $x = \cos\theta + i\sin\theta$ $\therefore \frac{1}{x} = \cos\theta - i\sin\theta$

& $x + \frac{1}{x} = 2\cos\theta$ & $x - \frac{1}{x} = 2i\sin\theta$

$x^n = \cos n\theta + i\sin n\theta$ & $\frac{1}{x^n} = \cos n\theta - i\sin n\theta$

$\therefore x^n + \frac{1}{x^n} = 2\cos n\theta$ & $x^n - \frac{1}{x^n} = 2i\sin n\theta$

(2M)

Now by Binomial thm

$(2i\sin\theta)^7 = \left(x - \frac{1}{x}\right)^7$

(1M)

$$(2i\sin\theta)^7 = x^7 - 7x^6\cdot\frac{1}{x} + 21x^5\cdot\frac{1}{x^2} - 35x^4\cdot\frac{1}{x^3} + 35x^3\cdot\frac{1}{x^4}$$
$$- 21x^2\cdot\frac{1}{x^5} + 7x\cdot\frac{1}{x^6} - \frac{1}{x^7}.$$

$$(2i\sin\theta)^7 = x^7 - 7x^5 + 21x^3 - 35x + \frac{35}{x} - \frac{21}{x^3} + \frac{7}{x^5} - \frac{1}{x^7}.$$

$$2^7 i^7 \sin^7\theta = \left(x^7 - \frac{1}{x^7}\right) - 7\left(x^5 - \frac{1}{x^5}\right) + 21\left(x^3 - \frac{1}{x^3}\right) - 35\left(x - \frac{1}{x}\right)$$

$$2^7\cdot i^7 \sin^7\theta = 2i\sin 7\theta - 7(2i\sin 5\theta) + 21(2i\sin 3\theta) - 35(2i\sin\theta).$$

$$-2^6 \sin^7\theta = \sin 7\theta - 7\sin 5\theta + 2\sin 3\theta - 35\sin\theta.$$ (2m)

$$\boxed{\sin^7\theta = -\frac{1}{2^6}\left(\sin 7\theta - 7\sin 5\theta + 2\sin 3\theta - 35\sin\theta\right)}$$

Q.6 b) If α & β are roots of the equation $x^2 - 2x + 4 = 0$

Prove that, $\alpha^n + \beta^n = 2^{n+1}\cos\frac{n\pi}{3}$.

→ Since $x^2 - 2x + 4 = 0$

Solving the eqn, $x = \frac{-(-2) \pm \sqrt{4 - 4(1)(4)}}{2}$

$$\Rightarrow x = \frac{2 \pm \sqrt{-12}}{2} = \frac{2 \pm 2\sqrt{3}\,i}{2}$$ (2m)

$$\Rightarrow \boxed{x = 1 \pm \sqrt{3}\,i}$$

Let $\alpha = 1 + \sqrt{3}\,i$ & $\beta = 1 - \sqrt{3}\,i$

$$\alpha = 2\left[\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right], \quad \beta = 2\left[\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right]$$

$$\therefore \alpha^n + \beta^n = 2^n\left[\cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3} + \cos\frac{n\pi}{3} - i\sin\frac{n\pi}{3}\right]$$ (2m)

$$\Rightarrow \alpha^n + \beta^n = 2^n\cdot\left(2\cos\frac{n\pi}{3}\right)$$

$$= 2^{n+1}\cdot\cos\frac{n\pi}{3}.$$ (1m)

Q. 7a) Prove that $\log\left(\frac{a+ib}{a-ib}\right) = 2i\tan^{-1}\left(\frac{b}{a}\right)$

Hence evaluate $\cos\left[i\log\left(\frac{a+ib}{a-ib}\right)\right]$.

$\rightarrow$ Putting $a = r\cos\theta$, $b = r\sin\theta \Rightarrow \theta = \tan^{-1}(b/a)$ (1M)

$$\log\left[\frac{a+ib}{a-ib}\right] = \log\left[\frac{r(\cos\theta + i\sin\theta)}{r(\cos\theta - i\sin\theta)}\right]$$

$$= \log\left[e^{i\theta} \div e^{-i\theta}\right]$$

$$= \log e^{2i\theta}$$

$$= 2i\theta$$ (2M)

$$= 2i\tan^{-1}(b/a).$$

Thus, $\cos\left[i\log\left(\frac{a+ib}{a-ib}\right)\right] = \cos\left[i(2i\theta)\right]$ (1M)

$$= \cos 2\theta$$

$$= \frac{1-\tan^2\theta}{1+\tan^2\theta}$$

$$= \frac{1-(b/a)^2}{1+(b/a)^2}$$ (1M)

$$= \frac{a^2-b^2}{a^2+b^2}.$$

Q.7 b) If $\cosh(u+iv) = x+iy$ P.T.

① $\frac{x^2}{\cosh^2 u} + \frac{y^2}{\sinh^2 u} = 1$ ② $\frac{x^2}{\cos^2 v} - \frac{y^2}{\sin^2 v} = 1$

$\rightarrow$ $\cosh(u+iv) = x+iy$.

$\Rightarrow \cosh u \cdot \cos v + i\sinh u \cdot \sin v = x+iy$ (1M)

Equating real & imaginary parts,

$x = \cosh u \cdot \cos v$; $y = \sinh u \sin v$.

to prove, $\dfrac{x^2}{\cosh^2 u} + \dfrac{y^2}{\sinh^2 u} = 1$

$$LHS = \frac{\cosh^2 u \cdot \cos^2 v}{\cosh^2 u} + \frac{\sinh^2 u \cdot \sin^2 v}{\sinh^2 u}$$

$$= \cos^2 v + \sin^2 v$$

$$= 1$$ (2m)

$= RHS$.

To prove $\dfrac{x^2}{\cos^2 v} - \dfrac{y^2}{\sin^2 v} = 1$

$$LHS = \frac{\cosh^2 u \cdot \cos^2 v}{\cos^2 v} - \frac{\sinh^2 u \cdot \sin^2 v}{\sin^2 v}$$

$$= \cosh^2 u - \sinh^2 u$$ (2m)

$$= 1$$

$= RHS$.

Q. 8 a) If $x = \cos[\log(y^{1/m})]$ then prove that

$(1-x^2) y_{n+2} - (2n+1) x y_{n+1} - (m^2+n^2) y_n = 0$.

1

$\rightarrow$ $x = \cos[\log y^{1/m}]$

$\Rightarrow \cos^{-1} x = \log y^{1/m}$

$\Rightarrow y^{1/m} = e^{\cos^{-1} x}$

$\Rightarrow y = e^{m\cos^{-1} x}$ —— ① (2m)

diff. w.r.t. x,

$$y_1 = e^{m\cos^{-1} x} \cdot \frac{-m}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2}\, y_1 = -m\, e^{m\cos^{-1} x}$$

$\Rightarrow \sqrt{1-x^2}\, y_1 = -my$ (1m)

Squaring

$(1-x^2)\, y_1^2 = m^2 y^2$

$(1-x^2) y_1^2 - m^2 y^2 = 0$

diff. w.r.t. x,

$(1-x^2) \cdot 2y_1 y_2 - 2x y_1^2 - m^2\, 2y y_1 = 0$

$(1-x^2) y_2 - x y_1 - m^2 y = 0$ (1m)

taking nth derivative.

$\left[(1-x^2) y_2\right]_n - \left[x y_1\right]_n - m^2 y_n = 0$

Applying Leibnitz thm.

$(1-x^2) y_{n+2} + n(-2x) y_{n+1} + \frac{n(n-1)}{2!}(-2) y_n - \left[x y_{n+1} + n y_n\right] = m^2 y_n$

$\Rightarrow \underline{(1-x^2) y_{n+2} - (2n+1) x y_{n+1} - (n^2+m^2) y_n = 0.}$ (1m)

Q. 8 b). If $u = \cos^{-1}\left[\frac{x+y}{\sqrt{x}+\sqrt{y}}\right]$ P.T. $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{2}\cot u$.

$\rightarrow \quad u = \cos^{-1}\left[\frac{x+y}{\sqrt{x}+\sqrt{y}}\right]$

Let $z = \cos u = \frac{x+y}{\sqrt{x}+\sqrt{y}}$

$\Rightarrow \quad z = \cos u = \frac{x\left[1 + y/x\right)}{\sqrt{x}\left[1+\sqrt{y/x}\right]} = x^{+1/2}\phi(y/x)$ (2m)

$\Rightarrow$ z is a homogeneous f^n of degree $1/2$ (1m)

$\therefore$ By Euler's thm,

$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = n \cdot z$ (1m)

$\Rightarrow x\left[\frac{dz}{du}\cdot\frac{\partial u}{\partial x}\right) + y\left(\frac{dz}{du}\cdot\frac{\partial u}{\partial y}\right) = \frac{1}{2}\cdot\cos u.$

$\Rightarrow \boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{2}\cot u}$ . . . $\left\{dz/du = -\sin u\right\}$ (1m)

Q. 9 a) If $u = f[e^{x-y}, e^{y-z}, e^{z-x}]$ then P.T.

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$$

$\rightarrow$ let $e^{x-y} = R$, $e^{y-z} = S$, $e^{z-x} = T$

$\Rightarrow u = f[R, S, T]$ (1m)

$\Rightarrow u \rightarrow R, S, T \rightarrow x, y, z.$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial R}\cdot\frac{\partial R}{\partial x} + \frac{\partial u}{\partial S}\cdot\frac{\partial S}{\partial x} + \frac{\partial u}{\partial T}\cdot\frac{\partial T}{\partial x}$$

(1m)

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial R}(e^{x-y}) + \frac{\partial u}{\partial S}\cdot(0) + \frac{\partial u}{\partial T}\cdot(-e^{z-x}) \quad \text{--- (1)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial R}\cdot\frac{\partial R}{\partial y} + \frac{\partial u}{\partial S}\cdot\frac{\partial S}{\partial y} + \frac{\partial u}{\partial T}\cdot\frac{\partial T}{\partial y} = \frac{\partial u}{\partial R}(-e^{x-y}) + \frac{\partial u}{\partial S}(e^{y-z}) + \frac{\partial u}{\partial T}(0) \quad \text{--- (2)} \quad \text{(1m)}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial R}\cdot\frac{\partial R}{\partial z} + \frac{\partial u}{\partial S}\cdot\frac{\partial S}{\partial z} + \frac{\partial u}{\partial T}\cdot\frac{\partial T}{\partial z} = \frac{\partial u}{\partial R}(0) + \frac{\partial u}{\partial S}\cdot(-e^{y-z}) + \frac{\partial u}{\partial T}(e^{z-x}) \quad \text{--- (3)} \quad \text{(1m)}$$

adding (1) (2) & (3) (1m)

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

Q. 9 b) Find the stationary values of x^3+y^3-3axy, $(a>0)$

$\rightarrow$ $f(x,y) = x^3+y^3-3axy$

$f_x = 3x^2-3ay$, $f_y = 3y^2-3ax$, $r = f_{xx} = 6x$

$s = f_{xy} = -3a$, $t = f_{yy} = 6y$ (2m)

$\rightarrow$ Let $f_x = 0$ & $f_y = 0$ solve as simultaneous eq^ns.

$x^2-ay=0$ & $y^2-ax=0$

$\Rightarrow$ $y = x^2/a$ put in 2nd eq^n

$$\frac{x^4}{a^2} - ax = 0 \Rightarrow x(x^3-a^3) = 0 \quad \text{(1m)}$$

$$\Rightarrow \boxed{x=0 \text{ or } x=a}$$

here $x=0$ or $x=a$.

When $x=0, y=0$ & when $x=a, y=a$

$(0,0)$ & (a,a) are stationary points,

i) for $(0,0)$ i.e. $x=0, y=0$

$r=f_{xx}=0$, $s=f_{xy}=-3a$ & $t=f_{yy}=0$

hence $rt-s^2=0-9a^2<0$ hence we reject this pair. (1m)

(ii) For (a,a).

$r=f_{xx}=6a$, $S=f_{xy}=-3a$, $t=f_{yy}=6a$

$\therefore rt-s^2=36a^2-9a^2=27a^2>0$

$\therefore f(x,y)$ is stationary at $x=a, y=a$

$\therefore$ But $r=f_{xx}=6a>0$ as $a>0$

$\therefore f(x,y)$ is minimum at $x=a, y=a$

Putting $x=a, y=a$ in $f(x,y)=x^3+y^3-3axy$ the

minimum value of $f(x,y)=a^3+a^3-3a^3=\underline{\underline{-a^3}}$ (1m)